

RICCI FLOW ON MODIFIED RIEMANN EXTENSIONS

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ABSTRACT. We study the properties of Modified Riemann extensions evolving under Ricci flow. We obtain the necessary and sufficient condition for modified Riemann extension under Ricci flow to stay as modified Riemann extension. We also discuss the properties of the curvature tensors under Ricci flow.

1. INTRODUCTION

Ricci flow and the evolution equations of the Riemannian curvature tensor were initially introduced by Richard Hamilton[4] and was later studied to a large extent by Perelman[12],[13],[14], Cao, Zhu[11], John Morgan, Gang Tian[15] and others. Indeed, the theory of Ricci flow has been used to prove the geometrization and Poincare conjectures[11]. However not much work has been done on Ricci flows on Modified Riemann extensions. The Ricci flow equation is the evolution equation $\frac{\partial g_{il}}{\partial t} = -2R_{il}$ where g_{il} and R_{il} are the metric and Ricci tensor components respectively. As flow progresses the metric changes and hence the properties related to it.

Patterson and Walker[5] have defined Riemann extensions and showed how a Riemannian structure can be given to the $2n$ dimensional tangent bundle of an n - dimensional manifold with given non-Riemannian

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structure. Riemann extension is an embedding of a manifold M in a manifold M' , the embedding being carried out in such a way that the geodesic equations are preserved up to the base space. The Riemann extension of Riemannian or non-Riemannian spaces can be constructed with the help of the Christoffel coefficients Γ_{jk}^i of corresponding Riemann space or with connection coefficients Π_{jk}^i in the case of the space of affine connection[9]. The theory of Riemann extensions has been extensively studied by Afifi[11]. Though the Riemann extensions is rich in geometry, here in our discussions, the modified Riemann extensions fit naturally in to the frame work. Modified Riemann extensions were introduced in [1] and [2] and their properties we list briefly in the next section.

Here in this paper we discuss some interesting properties satisfied by curvature tensors under the influence of Ricci flow on modified Riemann extensions. We give a brief introduction to modified Riemann Extensions[3] in section 2. In Section 3 we find the rate of change of concircular, conharmonic and conformal curvature tensors under Ricci flow. Ricci flow on modified Riemann extensions are discussed in Section 4.

2. PRELIMINARIES

Let (M, g) be a n -dimensional Riemannian manifold. Then Ricci flow is the evolution of the metric given by

$$\frac{\partial g_{il}}{\partial t} = -2R_{il}, \quad (2.1)$$

where g_{il} is the metric component and R_{il} is the component of the Ricci curvature tensor.

For a time dependent metric under Ricci flow, the evolution equations for Riemann curvature tensor, Ricci tensor and scalar curvature are given by[4],

$$\begin{aligned} \frac{\partial R_{ijkl}}{\partial t} = & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ & - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}), \end{aligned} \quad (2.2)$$

$$\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj} \quad (2.3)$$

and

$$\frac{\partial R}{\partial t} = \Delta R + 2g^{ij}g^{kl}R_{ik}R_{jl}, \quad (2.4)$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and R_{ijkl} , R_{ij} , R are the Riemannian curvature tensor, Ricci tensor and scalar curvature respectively.

Let ∇ be a torsion-free affine connection of M . The modified Riemann extension of (M, ∇) is the cotangent bundle T^*M equipped with a metric \bar{g} whose local components given by

$$\bar{g}_{ij} = -2\omega_l \Gamma_{ij}^l + c_{ij}, \quad \bar{g}_{ij^*} = \delta_i^j, \quad \bar{g}_{i^*j} = \delta_i^j \text{ and } \bar{g}_{i^*j^*} = 0.$$

The contravariant components are

$$\bar{g}^{ij} = 0, \quad \bar{g}^{ij^*} = \delta_i^j, \quad \bar{g}^{i^*j} = \delta_i^j \text{ and } \bar{g}^{i^*j^*} = 2\omega_l \Gamma_{ij}^l - c_{ij}$$

for i, j ranging from 1 to n and i^*, j^* ranging from $n+1$ to $2n$,

where ω_l are extended coordinates and c_{ij} is a $(0, 2)$ tensor on M .

The Christophel symbols are given by,

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl} \left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij} \right). \quad (2.5)$$

We note these results for the extended space, $\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k$,

$$\bar{\Gamma}_{ij}^{k^*} = \omega_l R_k^l j^i + \frac{1}{2}(\nabla_i c_{jk} + \nabla_j c_{ik} - \nabla_k c_{ij}),$$

$$\bar{\Gamma}_{i^*j}^k = 0, \quad \bar{\Gamma}_{i^*j}^{k^*} = -\Gamma_{jk}^i, \quad \bar{\Gamma}_{i^*j^*}^k = 0, \quad \bar{\Gamma}_{i^*j^*}^{k^*} = 0$$

The components of the Riemann curvature tensor of the extended space is given by

$$R_{ijk}^l = \frac{\partial}{\partial x^i} \Gamma_{jk}^l - \frac{\partial}{\partial x^j} \Gamma_{ik}^l + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m. \quad (2.6)$$

For the extended space ,

$$\begin{aligned} \bar{R}_{jkl}^i &= R_{jkl}^i, \\ \bar{R}_{jkl}^{i*} &= \omega_a (\nabla_j R_{ilk}^a - \nabla_k R_{ilj}^a) + \frac{1}{2} [\nabla_j (\nabla_l c_{ki} - \nabla_i c_{kl}) - \nabla_k (\nabla_l c_{ji} - \nabla_i c_{jl}) - \\ & R_{jkl}^m c_{mi} - R_{jki}^m c_{lm}], \quad \bar{R}_{j*kl}^i = R_{ilk}^j, \quad \bar{R}_{jk*l}^{i*} = -R_{ilk}^k, \quad \text{and} \quad \bar{R}_{jkl*}^{i*} = R_{kji}^l. \end{aligned}$$

The others are zero. i^*, j^*, k^*, l^* ranges from $n+1$ to $2n$. We lower the index in the middle position, to get

$$R_{ijkl} = g_{mk} R_{ijl}^m. \quad (2.7)$$

It may be noted by simple calculation that $\bar{R}_{i*jk*l} = 0$ which we require later on. Further, $\bar{R}_{ij} = R_{ij} + R_{ji}$, $\bar{R}_{i*j} = 0$ and $\bar{R}_{i*j*} = 0$.

3. EVOLUTION

The Ricci flow is given by equation(2.1). As time progresses the metric evolves and hence the properties depending on the metric change. Under Ricci flow, the rate of change of conformal curvature depends on the difference of conharmonic and Riemannian curvature tensors. The concircular curvature tensor is given by,

$$C_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)} [g_{il} g_{jk} - g_{jl} g_{ik}]. \quad (3.1)$$

The conharmonic curvature tensor is given by

$$L_{ijkl} = R_{ijkl} - \frac{1}{n-2} [g_{jk} R_{il} + g_{il} R_{jk} - g_{ik} R_{jl} - g_{jl} R_{ik}]. \quad (3.2)$$

Under Ricci flow, we give a relation between conformal tensor and conharmonic tensor.

Theorem 3.1. *For a manifold with non zero scalar curvature under Ricci flow, the rate of change of concircular tensor is related to con-harmonic tensor by*

$$\frac{\partial}{\partial t} \left(\frac{C_{ijkl} - R_{ijkl}}{R} \right) = \frac{2(n-2)}{n(n-1)} [R_{ijkl} - L_{ijkl}]. \quad (3.3)$$

Proof. Differentiating (3.1) we get,

$$\begin{aligned} \frac{\partial C_{ijkl}}{\partial t} &= \frac{\partial R_{ijkl}}{\partial t} - \frac{1}{n(n-1)} [g_{il}g_{jk} - g_{jl}g_{ik}] \frac{\partial R}{\partial t} \\ &\quad - \frac{R}{n(n-1)} \left[\frac{\partial g_{il}}{\partial t} g_{jk} + \frac{\partial g_{jk}}{\partial t} g_{il} - \frac{\partial g_{ik}}{\partial t} g_{jl} - \frac{\partial g_{jl}}{\partial t} g_{ik} \right]. \end{aligned} \quad (3.4)$$

$$\begin{aligned} \frac{\partial C_{ijkl}}{\partial t} - \frac{\partial R_{ijkl}}{\partial t} &= - \frac{1}{n(n-1)} [g_{il}g_{jk} - g_{jl}g_{ik}] \frac{\partial R}{\partial t} \\ &\quad + \frac{R}{n(n-1)} [2R_{il}g_{jk} + 2R_{jk}g_{il} - 2R_{ik}g_{jl} - 2R_{jl}g_{ik}]. \end{aligned} \quad (3.5)$$

But

$$[g_{il}g_{jk} - g_{jl}g_{ik}] = \frac{n(n-1)}{R} (R_{ijkl} - C_{ijkl}) \quad (3.6)$$

and

$$R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} = (n-2)(R_{ijkl} - L_{ijkl}). \quad (3.7)$$

Substituting (3.6) and (3.7) in (3.5) we get

$$\frac{\partial C_{ijkl}}{\partial t} - \frac{\partial R_{ijkl}}{\partial t} = - \frac{1}{R} (R_{ijkl} - C_{ijkl}) \frac{\partial R}{\partial t} + \frac{2(n-2)R}{n(n-1)} (R_{ijkl} - L_{ijkl}). \quad (3.8)$$

$$R \frac{\partial}{\partial t} (C_{ijkl} - R_{ijkl}) - (C_{ijkl} - R_{ijkl}) \frac{\partial R}{\partial t} = \frac{2(n-2)R^2}{n(n-1)} (R_{ijkl} - L_{ijkl}). \quad (3.9)$$

Therefore,

$$\frac{\partial}{\partial t} \left(\frac{C_{ijkl} - R_{ijkl}}{R} \right) = \frac{2(n-2)}{n(n-1)} [R_{ijkl} - L_{ijkl}]. \quad (3.10)$$

□

Example 1. Let M be a manifold with a space of constant curvature with $K = \frac{1}{1-n}$. Then evolution of the metric under Ricci flow is given by $g_{ij}(t) = g_{ij}(0)e^{-2t}$ and $R_{ijkl}(t) = R_{ijkl}(0)e^{-4t}$. Further, $C_{ijkl} - R_{ijkl} = -\frac{R}{n}R_{ijkl}$ and $L_{ijkl} - R_{ijkl} = \frac{2(n-1)}{n-2}R_{ijkl}$. Substituting this in equation (3.10) the above result is verified.

For a Riemannian manifold the Weyl conformal tensor is given by,

$$\begin{aligned} W_{ijkl} = & R_{ijkl} - \frac{1}{n-2}(g_{jk}R_{il} - g_{ik}R_{jl} + g_{il}R_{jk} - g_{jl}R_{ik}) \\ & + \frac{R}{(n-1)(n-2)}(g_{il}g_{jk} - g_{jl}g_{ik}). \end{aligned} \quad (3.11)$$

Equations (3.1), (3.2) and (3.11) can be combined to form,

$$(W_{ijkl} - L_{ijkl}) = -\frac{n}{n-2}(C_{ijkl} - R_{ijkl}). \quad (3.12)$$

Theorem 3.2. *For a n -manifold under Ricci flow,*

$$\frac{\partial}{\partial t} \left(\frac{W_{ijkl} - L_{ijkl}}{R} \right) = \frac{2}{n-1}(L_{ijkl} - R_{ijkl}). \quad (3.13)$$

Proof. Differentiating equation 3.12 with respect to 't' and using theorem 3.2 the result follows. □

4. EXTENSIONS

For modified Riemann Extensions, since the scalar curvature vanishes, the concircular curvature tensor is same as the Riemannian curvature tensor. Further the conharmonic curvature tensor is equal to the conformal curvature tensor.

Ricci flow on modified Riemann extensions is the evolution of metric such that the class of metrics obtained under Ricci flow can be expressed as modified Riemann extensions of a base metric. We prove the following results for Ricci flow on modified Riemann extensions.

Lemma 4.1. *Laplacian of Ricci tensor is zero on modified Riemann extension.*

Proof. Laplacian of Ricci tensor is given by,

$$\Delta R_{ij} = g^{kl} R_{ij:k:l}. \quad (4.1)$$

But,

$$\begin{aligned} g^{kl} R_{ij:k:l} = & g^{kl} R_{ij,k,l} - g^{kl} \Gamma_{jk,l}^{\alpha} R_{\alpha i} - g^{kl} \Gamma_{jk}^{\alpha} R_{\alpha i,l} - g^{kl} \Gamma_{ik,l}^{\alpha} R_{\alpha j} \\ & - g^{kl} \Gamma_{ik}^{\alpha} R_{\alpha j,l} - g^{kl} \Gamma_{il}^{\alpha} R_{\alpha j,k} + g^{kl} \Gamma_{il}^{\alpha} \Gamma_{k\alpha}^{\beta} R_{\beta j} + g^{kl} \Gamma_{il}^{\alpha} \Gamma_{jk}^{\beta} R_{\beta\alpha} \\ & - g^{kl} \Gamma_{jl}^{\alpha} R_{i\alpha,k} + g^{kl} \Gamma_{jl}^{\alpha} \Gamma_{\alpha k}^{\beta} R_{i\beta} + g^{kl} \Gamma_{jl}^{\alpha} \Gamma_{ik}^{\beta} R_{\beta\alpha} - g^{kl} \Gamma_{kl}^{\alpha} R_{ij,\alpha} \\ & + g^{kl} \Gamma_{kl}^{\alpha} \Gamma_{i\alpha}^{\beta} R_{\beta j} + g^{kl} \Gamma_{kl}^{\alpha} \Gamma_{j\alpha}^{\beta} R_{i\beta}. \end{aligned} \quad (4.2)$$

From the properties of extended metric components we have, g^{kl} to be non zero atleast one of k or l must be greater than n . Suppose $k > n$, then $R_{ij,k} = 0$. Also $R_{\alpha i} \neq 0$ only when $\alpha < n$ and $i < n$. But if $\alpha \leq n$ then $\Gamma_{jk}^{\alpha} = 0$ since $k > n$. Similar argument makes all the terms on the right side of the equation to vanish. If $l > n$ then again $R_{ij,k,l}$ vanishes since $R_{ij,k}$ is a function of first n coordinates. Also, since Christoffel symbols are preserved by extension, $\Gamma_{jk,l}^{\alpha}$ vanishes. Hence the result. \square

Theorem 4.3. *The Ricci curvature tensor is independent of time for Ricci flow on modified Riemann extensions.*

Proof. Let M be an n -dimensional manifold. The rate of change of Ricci tensor is given by

$$\frac{\partial R_{ik}}{\partial t} = \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}. \quad (4.3)$$

for i, k greater than n , $R_{ik} = 0$. It is sufficient to prove for i, k ranging from 1 to n . For g^{pr} and g^{qs} to be non zero, either $p > n$ or $r > n$ and $q > n$ or $s > n$. Suppose $p > n$ and $q > n$. Then as discussed earlier $R_{piqk} = 0$. If $s > n$ or $r > n$ then $R_{rs} = 0$. Thus $2g^{pr}g^{qs}R_{piqk}R_{rs} = 0$. Now g^{pq} is non zero for $p > n$ or $q > n$. But if $p > n$, $R_{pi} = 0$ and similarly if $q > n$, $R_{qk} = 0$. Hence the result. \square

It must be noted here that the flow is not on the base manifold but on the extended space. We have proved the necessary and sufficient conditions for Modified Riemann extension under Ricci flow to stay as modified Riemann extensions.

We can restate the result in terms of metric.

Theorem 4.4. *Ricci flow on modified Riemann Extensions is linear.*

Proof. Under Ricci flow on modified Riemann Extensions, the Ricci tensor is time invariant. Hence on solving (2.1) we get

$$g_{jk}(t) = R_{jk}t + g_{jk}(0). \quad (4.4)$$

Thus the metric is linearly varying with time. \square

Example 2. *modified Riemann extension of Schwarzschild metric has vanishing Ricci tensor and hence remains a trivial example.*

Example 3. Consider the hyperbolic metric $ds^2 = \frac{1}{y^2}dx^2 + \frac{1}{y^2}dy^2$. The modified Riemannian extension of this is

$$ds^2 = \frac{-4P}{y}dx^2 - \frac{8P}{y}dxdy + \frac{4Q}{y}dy^2 + 2dxdp + 2dQdy. \quad (4.5)$$

where $c_{ij} = 0$.

Then $R_{11} = \frac{2}{y^2} = R_{22}$ and rest of the components equal to zero. Thus $g_{11} = \frac{2}{y^2}t + \frac{-4Q}{y}$ and $g_{22} = \frac{-4Q}{y} + \frac{2}{y^2}t$ with rest of the components independent of time which are the required class of metric components.

Theorem 4.5. *For modified Riemann Extensions under Ricci flow, the rate of change of extended components of Weyl conformal tensor is the same as the rate of change of extended components of Riemann curvature tensor.*

Proof. For the extended space, the Weyl conformal tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}). \quad (4.6)$$

Differentiating partially with respect to 't' we get,

$$\begin{aligned} \frac{\partial W_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{1}{n-2} & \left(\frac{\partial g_{ik}}{\partial t} R_{jl} + g_{ik} \frac{\partial R_{jl}}{\partial t} - \frac{\partial g_{il}}{\partial t} R_{jk} - g_{il} \frac{\partial R_{jk}}{\partial t} \right. \\ & \left. - \frac{\partial g_{jk}}{\partial t} R_{il} - g_{jk} \frac{\partial R_{il}}{\partial t} + \frac{\partial g_{jl}}{\partial t} R_{ik} + g_{jl} \frac{\partial R_{ik}}{\partial t} \right). \end{aligned} \quad (4.7)$$

Using previous theorem and (2.1) and rearranging we get

$$\frac{\partial W_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{4}{n-2}(R_{il}R_{jk} - R_{ik}R_{jl}). \quad (4.8)$$

Here again for any two of i, j, k, l greater than n the Ricci components are zero. In particular for all of them greater than n , we get the above result. \square

CONCLUSION

We have found the necessary and sufficient conditions for the the modified Riemann extension under Ricci flow evolving to obtain a class of metrics which again are modified Riemann extensions.

REFERENCES

- [1] E. Calviño-Louzao, E. García-Río, P. Gilkey and A. Vazquez-Lorenzo, The geometry of modified Riemannian extensions. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 2009., Vol. 465, no. 2107, 20232040.
- [2] E. Calviño-Louzao, E. García-Río and R. Vázquez-Lorenzo, Riemann extensions of torsionfree connections with degenerate Ricci tensor. *Can. J. Math.* (2010)., Vol. 62, No. 5, 10371057.
- [3] Aydin Gezer, Lokman Bilen, and Ali Cakmak Properties of modified Riemannian extensions. *arXiv:1305.4478v2 [math.DG]* 26 May 2013.
- [4] Richard S. Hamilton., Three-manifolds with positive Ricci curvature. *J. Differential. Geom.*, 1982., Vol 17., 255-306.
- [5] Paterson E.M and Walker., Riemann extensions, *Quart. J. Math. Oxford*, 1952., 3, 19-28.
- [6] A. G. Walker., Canonical form for a Riemannian space with parallel field of null planes, *Quart. J. Math. Oxford*, 1950., Vol. 1, 67-69.
- [7] Luther Pfahler Eisenhart., Fields of Parallel Vectors in Riemannian Space., *Annals of Mathematics*, 1938, Vol. 39, No. 2, 316-321.
- [8] Oldřich Kowalski and Masami Sekizawa., The Riemann extensions with cyclic parallel Ricci tensor., *Math. Nachr.*, 2014, Vol. 287, No. 8-9, 955-961.
- [9] V. S. Dryuma Four dimensional Einstein spaces on six dimensional Ricci flat base spaces. *arXiv:gr-qc/0601051*, 2006., Vol 12 .No. 1.
- [10] Z. Afifi, Riemann extension of affine connected spaces., *Quart. J. Math. Oxford* 1954., Vol. 2, 5 , 312-30.
- [11] Huai-Dong Cao and Xi-Ping Zhua, complete proof of the Poincare and geometrization conjectures application of the Hamilton-Perelman theory of the Ricci flow., *Asian J. Math.*, 2006., Vol. 10, No. 2, 165492.
- [12] Grigory Parelman, The Entropy Formula for the Ricci Flow and its Geometric Applications, *arXiv:math/0211159v1 [math.DG]* 11 Nov 2002.
- [13] Grigory Parelman, Ricci Flow with Surgery on Three-Manifolds, *arXiv:math/0303109v1 [math.DG]* 10 Mar 2003.
- [14] Grigory Parelman, Finite Extinction Time for the Solutions to the Ricci Flow on certain Three-Manifolds, *arXiv:math/0307245v1 [math.DG]* 17 Jul 2003.

- [15] John Morgan and Gang Tian, Ricci Flow and the Poincaré Conjecture, arXiv:math.DG/0607607 v1 25 Jul 2006.

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